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Contract No.: DA-31-124-ARO-D-462

ITERATIVE METHODS FOR BEST APPROXIMATE SOLUTIONS OF LINEAR INTEGRAL EQUATIONS OF THE FIRST AND SECOND KINDS

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MRC Technical Summary Report # 1117
September 1971

Received December 18, 1970

Madison, Wisconsin 53706

ABSTRACT

Least squares solutions of Fredholm and Volterra equations of the first and second kinds are studied using generalized inverses. The method of successive approximations, the steepest descent and the conjugate gradient methods are shown to converge to a least squares solution or to the least squares solution of minimal norm, both for integral equations of the first and second kinds.

An iterative method for matrices due to Cimmino is generalized to integral equations of the first kind and its convergence to the least squares solution of minimal norm is established.

ITERATIVE METHODS FOR BEST APPROXIMATE SOLUTIONS OF LINEAR INTEGRAL EQUATIONS OF THE FIRST AND SECOND KINDS

W. J. Kammerer[1] and M. Z. Nashed

1. Introduction.

Linear integral equations of the first and second kinds that have non-unique solutions or that have no solution at all arise in many settings. Physical problems may lead to such situations directly as in the case of the integral equation formulation of the interior Neumann problem for the Laplacian on a simply connected region with a smooth boundary (see, for instance, [13; pp. 341-344]). On the other hand, one is led to such situations via eigenvalue problems, as in the case of a nonhomogeneous integral equation of the second kind when the associated homogeneous equation has a nontrivial solution. In this case, if the prescribed function appearing in the integral equation satisfies the compatability condition of the Fredholm alternative, then we have an infinite number of solutions; otherwise we have no solution.

There are a number of cases in which one would like to find the solution of minimal norm to a non-uniquely solvable Fredholm or Volterra integral equation, or to seek least squares solutions when the integral equation in question does not have a solution in the classical sense.

The bulk of the work on iterative methods for solution of linear integral equations is devoted to equations which have unique solutions (see [48], [36], [34], [28], [31]; these references also contain relevant bibliographies). The main purpose of the present paper is to investigate best approximate solutions, i.e. solutions in the sense of least squares, and to establish the convergence of the method of successive approximations, the steepest descent and the conjugate gradient methods to best approxi-

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Sponsored by the United States Army under Contract No.: DA-31-124-ARO-D-462.

mate solutions of integral equations of the first and second kinds. We also generalize to integral equations of the first kind an iterative method for matrices due to Cimmino [5] and establish its convergence to a best approximate solution of minimal norm.

It is well known that the numerical solution of integral equations of the first kind leads to difficulties both for the Fredholm and the Volterra equations, since the solution does not depend continuously on the data. This instability of the integral equation of the first kind also carries over to the solutions of the algebraic system arising from discretization of the integral equation. Continuous dependence of solutions on the data can be brought about by taking the notion of a "solution" to mean a "least squares solution" and by restricting the class of admissible solutions in a suitable way, for instance by constraining the solution to have an apriori bound. Thus the notion of a least squares solution is part of a natural setting for ill-posed problems and lends itself more readily to mathematical programming, (see [8]). Moment discretizations and related aspects of smoothness and regularization in least squares solutions will be examined using generalized inverses in a subsequent note.

A numerical approach to obtain an approximate solution to a non-uniquely solvable Fredholm integral equation of the second kind has been studied recently by Atkinson [1]. First the equation is recast as a new uniquely solvable equation and then the integral operator is approximated using numerical integration. Our approach, in contrast, is iterative, and we do not stipulate that the equation is solvable since we seek solutions in the least squares sense. We carry parallel developments for the integral equations of the first and second kinds, using generalized inverses as a basic tool throughout.

2. Generalized Inverses and Linear Operator Equations of the First and Second Kinds.

Let X and Y be two Hilbert spaces over the same (real or complex) scalars, and let T be a bounded linear operator on X into Y, whose range is not necessarily closed. We denote the range of T by $R(T)$, the null space of T by $N(T)$ and the adjoint of T by T^*. For any subspace S of X or Y, we denote by $S^\perp$ the orthogonal complement of S and by $\bar{S}$ the closure of S. Then the following relations hold (see, for instance, Yosida [50])

$$(2.1) \qquad X = N(T) \oplus N(T)^\perp,$$

$$(2.2) \qquad Y = N(T^*) \oplus N(T^*)^\perp,$$

$$(2.3) \qquad \overline{R(T)} = N(T^*)^\perp, \ \overline{R(T^*)} = N(T)^\perp,$$

and the restriction of T to $N(T)^\perp$, denoted by $T|N(T)^\perp$, has an inverse which is necessarily not continuous. Let P denote the orthogonal projection of X onto $N(T)^\perp$ and let Q denote the orthogonal projection of Y onto $N(T^*)^\perp$. Then

$$(2.4) \qquad Tx = TPx \quad \text{for all } x \in X,$$

and

$$(2.5) \qquad T^*y = T^*Qy \quad \text{for all } y \in Y.$$

We associate with the linear operator equation

$$(2.6) \qquad Tx = y \quad \text{for } y \in Y,$$

the projectional equation

(2.7) $$Tx = Qy \quad \text{for } y \in Y.$$

Obviously, (2.7) is different from (2.6) only if $R(T)$ is not dense in Y, in which case the solvability of (2.7) does not necessarily imply the solvability of (2.6). For any $y \in R(T) \oplus R(T)^{\perp}$, (2.7) is solvable and the set S of all solutions is a closed convex subset of X; hence it contains a unique element of minimal norm. In this manner, we get a mapping which assigns to each $y \in R(T) \oplus R(T)^{\perp}$, the unique solution of minimal norm of (2.7). We call this mapping the generalized inverse of T and denote it by $T^{\dagger}$. We formalize this in the following definition.

Definition 2.1. Let $T : X \to Y$ be a bounded linear operator. The generalized inverse $T^{\dagger}$ of T is the mapping whose domain is $D(T^{\dagger}) = R(T) \oplus R(T)^{\perp}$, and $T^{\dagger}y = v \in X$, where $v \in S = \{x \in X : Tx = Qy\}$ and $\|v\| < \|u\|$ for all $u \in S$, $u \neq v$.

Note that $N(T^{\dagger}) = N(Q) = \overline{R(T)}^{\perp}$ and that $T^{\dagger}$ is linear but not necessarily bounded.

A natural question that arises is: What is the significance of a solution of (2.7) as far as (2.6) is concerned? To this end, we state the following proposition.

Proposition 2.1. For a fixed $y \in Y$, let $S = \{x \in X : Tx = Qy\}$ and $N = \{x \in X : T^{*}Tx = T^{*}y\}$. Then $S = N$.

Proof. Let $u \in S$, then $T^{*}Tu = T^{*}Qy = T^{*}y$ by (2.5). Conversely, if $v \in N$, then $T^{*}Tv = T^{*}y = T^{*}Qy$, which means that $Tv - Qy \in N(T^{*})$. But $Tv - Qy$ is also in $\overline{R(T)}$. Therefore, $Tv - Qy \in N(T^{*}) \cap \overline{R(T)} = \{0\}$.

Definition 2.2. An element $u \in X$ is called a least squares solution of the linear operator equation (2.6) if

$$\|Tu - y\| = \inf \{\|Tx - y\| : x \in X\}.$$

An element $\bar{x}$ is called a least squares solution of minimal norm of (2.6) if $\bar{x}$ is a least squares solution of (2.6) and $\|\bar{x}\| \leq \|u\|$ for all least squares solutions u of (2.6).

For a given $y \in Y$, the set of all least squares solutions of (2.6) coincides with the set N of all solutions of the "normal" equation $T^*Tx = T^*y$. Combining this observation with Proposition 2.1 and Definition 2.1, we arrive at the following proposition.

Proposition 2.2. Let T be a bounded linear operator on X into Y, whose range is not necessarily closed. The least squares solution $\bar{x}$ of minimal norm of the linear operator equation

$$Tx = y \qquad\qquad y \in \mathcal{D}(T^\dagger)$$

is given by $\bar{x} = T^\dagger y$. The set of all least squares solutions for each $y \in \mathcal{D}(T^\dagger)$ is given by $T^\dagger y \oplus N(T)$.

It follows easily from the definitions of P, Q, and $T^\dagger$ that $P = T^\dagger T$, Q is the continuous linear extension of $TT^\dagger$ to Y, and $T^\dagger \left[R(T)^\perp\right] = \{0\}$.

If the range of T is closed, then by the closed range theorem (see Yosida [50]), the range of T^* is also closed and one obtains from (2.3) and (2.2), $R(T) = N(T^*)^\perp$, $R(T^*) = N(T)^\perp$, and $Y = R(T) \oplus R(T)^\perp$. The generalized inverse in this case is defined on all of Y and is also bounded, and $Q = TT^\dagger$. The following proposition sheds more light on the generalized inverse of a bounded linear operator with closed range; see also [33].

Proposition 2.3. For a bounded linear operator $T : X \to Y$, the following statements are equivalent:

(a) T has a bounded generalized inverse.

(b) T has a bounded right inverse.

(c) The restriction of T to $N(T)^{\perp}$ has a bounded inverse.

(d) The operator $\{T|N(T)^{\perp}\}^{-1}$ has a bounded linear extension to all of Y so that its null space is $N(T^*)$.

(e) The quotient space $X/N(T)$ is isomorphic with $R(T)$.

(f) The number $\gamma(T)$ defined by $\gamma(T) = \sup\{\inf\{\|x\| : Tx = y\} : y \in R(T), \|y\| = 1\}$ is finite.

(g) $R(T)$ is closed.

(h) T is normally solvable in the sense of Hausdorff, i.e. for a given $y \in Y$, the necessary and sufficient condition for the equation $Tx = y$ to be solvable is that $y \in N(T^*)^{\perp}$.

(i) $\inf\{\|Tx - y\| : x \in X\}$ is attained for each $y \in Y$.

(j) All pseudoinverses of T are bounded. (A linear operator M with the property that $TMT = T$ is called a pseudoinverse of T.)

(k) There exists a unique operator $T^{\dagger} : Y \to X$ such that $T^{\dagger}TT^{\dagger} = T^{\dagger}$, $T^{\dagger}T = P$, and $TT^{\dagger} = Q$.

Remark 2.1. The concept of the generalized inverse of a linear operator plays a decisive role in the study of convergence of iterative methods for integral equations that are considered in the present paper. It is appropriate here to point out that historically notions of generalized or pseudo inverses appeared first in the context of analysis, rather than in the setting of matrices and algebraic problems. More specifically, the germ of these notions may be found in the celebrated paper of Fredholm [10], in the work of Hurwitz [20] on

pseudo-resolvents, the work of Hilbert [18], Elliott [9], Reid [37], and others on generalized Green's functions (see [39] for other references and a detailed discussion of this aspect of the history of generalized inverses).

The classical development of integral equations during the first half of this century is rich with instances of implicit ideas and concepts which led to fundamental abstractions in the setting of functional analysis. Many aspects of linear and nonlinear operator theory (for example, compact operators, spectral theory of Riesz-Schauder, gradient mappings, monotone operators, collectively compact operators, etc.) owe their origin or motivation to considerations of integral equations. In turn, when the various aspects of the theory of integral equations are re-examined in the light of such results of operator theory, usually new insight and results are obtained

The operator-theoretic framework in terms of generalized inverses is a natural setting for integral equations which have no solutions in the classical sense, or which have non-unique solutions. Generalized inverses have been recently used in connection with Green's functions and boundary value problems and other aspects of ordinary and partial differential equations by Reid [38], [39], Loud [29], [30], Wyler [49], Landesman [27], Kallina [21], Halany and Moro [14] and others.

Remark 2.2. The earliest work explicitly devoted to the study of generalized inverses of linear operators is that of Tseng [44], [45], [46] (see also [2] for a summary of some of Tseng's results and a comprehensive development of some aspects of generalized inverses, together with an extensive bibliography). Pseudoinverses of linear operators were also defined and used by Friedrichs [12], Hamburger [15], and Sheffield [40]. The generalized inverse of a continuous linear transformation with closed range has been studied by

Desoer and Whalen [7], Beutler [3], Petryshyn [35], Sheffield [40], Votruba [47] and others. Various (not necessarily equivalent) definitions of generalized inverses for the case when the range of the continuous operator is not closed as well as for unbounded closed operators have been proposed by Tseng [45], [46], Hestenes [16], Beutler [3], and others. Pseudoinverses for closed operators with closed range have been briefly considered also by Wyler [49] and Reid [39].

The definition of a generalized inverse that was introduced in this section is equivalent in the case when $R(T)$ is closed to the definitions in [7], [3], [35]. It has the advantages of focusing on the problem of solvability of the operator equation and of treating the cases when $R(T)$ is closed and when $R(T)$ is not closed in the same framework, thereby exhibiting the distinctive features of these two cases as far as least squares solutions are concerned.

In Proposition 2.3 we stated several characterizations of a bounded linear operator with closed range. The following proposition exhibits specific classes of such operators, which will be used in the analysis of integral operators in the following sections.

For any linear operator $T : X \to X$,

$$\{0\} \subset N(T) \subset N(T^2) \subset \cdots \subset N(T^k) \subset \cdots$$

and

$$X \supset R(T) \supset R(T^2) \supset \cdots \supset R(T^k) \supset \cdots .$$

We recall that T is said to have a <u>finite</u> <u>ascent</u> if for some nonnegative integer r, $N(T^r) = N(T^{r+k})$ for $k = 1,2, \ldots$. In this case, the smallest such integer is called the <u>ascent</u> of T. The <u>descent</u> of T is similarly defined as the smallest integer r for which $R(T^r) = R(T^{r+k})$ for $k = 1,2, \ldots$, assuming T is of finite

ascent. If T has a finite descent δ and a finite ascent α, then $\alpha = \delta$ and $X = R(T^{\alpha}) \oplus N(T^{\alpha})$ (see [43]). In connection with integral operators, we note that for every compact linear operator A on a normed linear space X into X, the operator $\lambda I - A$ has a finite ascent and descent if $\lambda \neq 0$. Also, if T is a normal operator, i.e. $T^*T = TT^*$, the ascent is either 0 or 1 (see [43], Theorem 6.2 F).

Proposition 2.4. The set of all bounded linear operators with closed range includes the following classes of operators:

(a) all operators which are bounded below, i.e., $\|Tx\| \geq m\,\|x\|$, $m > 0$, for all $x \in X$;

(b) all operators of the form $T = T_1 + T_2$, where $R(T_1)$ is closed and $R(T_2)$ is finite dimensional;

(c) all operators of the form $T = A - \lambda L$, $\lambda \neq 0$ where A is completely continuous (i.e., maps each bounded set into a compact set) and L has a bounded inverse;

(d) all continuous normal operators of finite descent.

Proof. Parts (a) and (b) are obvious. To prove (c) we recall [51] that for any $\epsilon > 0$, a completely continuous operator A can be decomposed in the form $A = A_1 + A_2$, where $\|A_1\| < \epsilon$ and $R(A_2)$ is finite dimensional. Take $\epsilon = \frac{\lambda}{\|L^{-1}\|}$. Then $T = (A_1 - \lambda L) + A_2$ and $\left\|\frac{L^{-1}A_1}{\lambda}\right\| < 1$. This implies that $A_1 - \lambda L$ is invertible on all of Y and hence has a closed range. Thus T has a closed range by part (b).

(d) Since T is assumed to be normal and of finite descent, its descent must be either zero or one. If the descent is zero, then $R(T) = X$ and hence closed. Suppose now that the descent of T is one, then $X = R(T) \oplus N(T)$. But we also have $X = N(T) \oplus N(T)^{\perp}$ since T is continuous. Thus $R(T)$ is closed if and only if

$R(T) = N(T)^{\perp}$. Let $x \in N(T)^{\perp}$, then $x = Tu + z$, $u \in X$, $z \in N(T)$, and $\langle x, v\rangle = 0$ for all $v \in N(T)$. But $\langle Tu, v\rangle = \langle u, T^{*}v\rangle = 0$ for $v \in N(T)$, since $N(T) = N(T^{*})$ for any normal operator. This implies that $\langle z, v\rangle = 0$ for all $v \in N(T)$. Setting $v = z$, we get $z = 0$. Hence $x = Tu + z = Tu$, i.e. $u \in R(T)$. This proves that $N(T)^{\perp} \subset R(T)$. The inclusion $R(T) \subset N(T)^{\perp}$ is obvious.

Proposition 2.5. A completely continuous linear operator $T : X \to Y$ does not have a closed range unless $R(T)$ is finite dimensional.

Proof. Suppose T is completely continuous and $R(T)$ is closed. Then T has a bounded generalized inverse $T^{\dagger}$ defined on all of Y. Thus $TT^{\dagger}$ being the composition of a completely continuous operator and a bounded operator is completely continuous. On the other hand, we have $TT^{\dagger} = Q$. Thus $Q|R(T) = I|R(T)$ is completely continuous, which implies that $R(T)$ is finite dimensional.

We recall also that the range of a completely continuous linear operator T is always separable and that $\sigma_p(T)$, the point spectrum of T, contains at most a countable set of points with zero the only possible accumulation point.

We now consider the linear operator equations

$$Ax - \lambda x = y \tag{2.8}$$

$$Ax = y \tag{2.9}$$

where y is a given element in a Hilbert space H and A is a completely continuous linear operator on H into H. (2.9) and (2.8) are usually referred to as equations of the first and second kinds, respectively, by analogy with integral equations. For any $\lambda \neq 0$, $R(A - \lambda I)$ is closed and we have from (2.1) - (2.3),

$$(2.10) \qquad H = R(A - \lambda I) \oplus N(A^* - \bar{\lambda} I)$$

$$= R(A^* - \bar{\lambda} I) \oplus N(A - \lambda I), \ \lambda \neq 0.$$

The criterion for solvability of (2.8) can be completely analyzed using the well known theorems of Fredholm - Riesz, which are based on (2.10), and the following relations for $\lambda \neq 0$ (see, for instance, [50], [51]):

$$\dim N(A - \lambda I) = \dim N(A^* - \bar{\lambda} I) < \infty$$

$$\dim R(A - \lambda I) = \dim R(A^* - \bar{\lambda} I)$$

It then follows that for a given $\lambda \neq 0$, (2.8) has a solution for all $y \in H$ if and only if $N(A - \lambda I) = \{0\}$. On the other hand, (2.8) has a solution for a given $y \in H$ and $\lambda \neq 0$ if and only if y is orthogonal to $N(A^* - \bar{\lambda} I)$. If $\lambda \neq 0$ is not an eigenvalue of the operator A, then $(A - \lambda I)^{-1}$ is bounded and $R(A - \lambda I) = H$. If $\lambda \neq 0$ is an eigenvalue of A, then $R(A - \lambda I)$ is a closed proper subspace of H.

For the solvability of (2.9) for a given $y \in H$, the condition $y \in N(A^*)^{\perp}$ is necessary but not sufficient, since the range of A is not closed unless it is finite dimensional (Proposition 2.5). The alternative theorem does not hold, and one does not get a decomposition theorem of H in terms of $R(A)$ and $N(A^*)$. On the other hand, one can determine the additional requirement that $y \in N(A^*)^{\perp}$ must satisfy in order for (2.9) to be solvable, in terms of the eigenvalues $\{\mu_n\}$ and the orthonormal eigenvectors $\{\phi_n\}$ of the operator AA^*, namely,

(2.11) $$\sum \frac{1}{\mu_n} |\langle y, \phi_n\rangle|^2 < \infty.$$

(See, for instance, [6], [42].)

In the present paper, we are primarily interested in the case when the above solvability criteria are not satisfied, so that (2.8) and (2.9) do not have solutions, and also in the case when these equations have an infinite number of solutions. We are interested in "best approximate solutions" of these equations or, more precisely, least squares solutions of minimal norm in the sense of Definition 2.2. The following theorem follows easily from the preceding propositions.

Theorem 2.1. Let A be a completely continuous linear operator on a Hilbert space H into H.

(a) For each $\lambda \neq 0$, the operator $A - \lambda I$ has a bounded generalized inverse $(A - \lambda I)^\dagger$ defined on all of H and $\hat{x} = (A - \lambda I)^\dagger y$ is the unique best approximate solution of minimal norm of (2.8) for each $y \in H$, i.e.,

$$\inf \{\|(A - \lambda I)x - y\| : x \in H\} = \|(A - \lambda I)\hat{x} - y\|$$

and $\|\hat{x}\| < \|u\|$ for all u such that

$$\|(A - \lambda I)\hat{x} - y\| = \|(A - \lambda I)u - y\|, \quad u \neq \hat{x}.$$

In particular if $\lambda \neq 0$ and $y \in R(A - \lambda I)$, then $\hat{x} = (A - \lambda I)^\dagger y$ is the unique solution of (2.8) of minimal norm. If $\lambda \neq 0$ is in the resolvent of A, then $(A - \lambda I)^\dagger = (A - \lambda I)^{-1}$ and (2.8) has a unique solution for each $y \in H$.

(b) The operator A has a generalized inverse $A^{\dagger}$ defined on $\mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}$. $A^{\dagger}$ is unbounded unless $\mathcal{R}(A)$ is finite dimensional. The linear operator equation (2.9) has a unique least squares solution for each $y \in \mathcal{D}(A^{\dagger})$. If also $y \in N(A^{*})^{\perp}$ and (2.11) holds, then $\hat{x} = A^{\dagger}y$ is the unique solution of (2.9) of minimal norm.

(c) The set of all least squares solutions of (2.8) for $\lambda \neq 0$ is given by $(A - \lambda I)^{\dagger}y \oplus N(A - \lambda I)$ for each $y \in H$. The set of all least squares solutions of (2.9) for each $y \in \mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp}$ is given by $A^{\dagger}y \oplus N(A)$.

Remark 2.3. The operators P and Q played a crucial role in the definition and development of a generalized inverse of a bounded linear operator between two Hilbert spaces. The definition can be extended easily to Banach spaces. We consider for instance the case of a continuous linear operator $T : X \to Y$, where X and Y are Banach spaces over the real or complex numbers, and T has a closed range. Let P_1 and P_2 be given projectors onto $N(T)$ and $\mathcal{R}(T)$ respectively. (By a projector P we mean as usual a continuous linear and idempotent ($P^2 = P$) operator.) The unique bounded linear operator $T^{\dagger}$ (which depends on P_1 and P_2) of Y into X satisfying $T^{\dagger}TT^{\dagger} = T^{\dagger}$, $T^{\dagger}T = I - P_1$, and $TT^{\dagger} = P_2$ is called the generalized inverse of T relative to the projectors P_1 and P_2. In the case of Hilbert spaces we have chosen P_1 and P_2 to be the orthogonal (equivalently self-adjoint) projectors. Although other choices are possible, they do not lead to the desirable connection with least squares solutions stated in Proposition 2.2.

Finally, we remark that in the case of a generalized inverse on a Banach space (whose norm is not induced by an inner product), $T^{\dagger}y$ is not necessarily a best approximate solution of $Tx = y$ for $y \in \mathcal{D}(T^{\dagger})$.

3. Best Approximate Solutions of Fredholm and Volterra Linear Integral Equations of the First and Second Kinds.

Throughout this section, the kernel K(s, t) is a function in $L_2\{[a, b] \times [a, b]\}$, i.e.,

$$\int_a^b \int_a^b |K(s, t)|^2 \, ds \, dt < \infty ,$$

and y is a given element in $L_2[a, b]$ with the usual inner product. For simplicity of notation, we take K(s, t) to be real; all the results hold for complex kernels as well with obvious modifications. Let $\mathfrak{K}$ and $\mathfrak{B}$ denote respectively the Fredholm and Volterra linear integral operators

$$\mathfrak{K}x = \int_a^b K(\cdot, t) \, x(t) \, dt \tag{3.1}$$

and

$$\mathfrak{B}x = \int_a^{\cdot} K(\cdot, t) \, x(t) \, dt . \tag{3.2}$$

$\mathfrak{K}$ and $\mathfrak{B}$ map $L_2[a, b]$ into itself and are completely continuous. We consider the Fredholm equations of the first and second kinds

$$\mathfrak{K}x = y , \tag{3.3}$$

$$x - \lambda \mathfrak{K}x = y , \tag{3.4}$$

and the Volterra equations of the first and second kinds

(3.5) $$\mathfrak{K}x = y ,$$

(3.6) $$x - \lambda\mathfrak{K}x = y .$$

A function $u \in L_2[a, b]$ is called a <u>best approximate solution</u> of (3.3) if it minimizes the functional $\|\mathfrak{K}x - y\|$ in the L_2-norm. An element is a best approximate solution if and only if

(3.7) $$\mathfrak{K}^*\mathfrak{K}u = \mathfrak{K}^*y$$

where

$$\mathfrak{K}^*x = \int_a^b K(t,\cdot)\, x(t)\, dt .$$

Equation (3.7) can be put in the form

(3.8) $$\int_a^b M(s, t)\, x(t)\, dt = y_1 ,$$

where $M(s, t) = \int_a^b K(r, s)\, K(r, t)\, dr$, and $y_1(t) = \int_a^b K(t, s)\, y(t)\, dt$. As pointed out in a more general setting in Section 2, (3.8) need not have solutions for all $y \in H$.

Similarly the problem of finding best approximate solutions of (3.4) is equivalent to solving the normal equation

(3.9) $$(I - \lambda(\mathfrak{K} + \mathfrak{K}^*) + \lambda^2\mathfrak{K}^*\mathfrak{K})x = (I - \lambda\mathfrak{K}^*)y,$$

or equivalently,

(3.10) $$\begin{cases} x(s) - \lambda \int_a^b k(s, t)\, x(t)\, dt = g(s), \text{ where} \\ k(s, t) = K(s, t) + K(t, s) - \lambda M(s, t), \text{ and } g(s) = y(s) - \lambda y_1(s) . \end{cases}$$

Note that the kernel k(s, t) is always symmetric and that λ appears quadratically is (3.10). Equation (3.10) always has solutions.

It is easy to show that the adjoint of the Volterra operator $\mathfrak{B}$ is given by

$$\mathfrak{B}^* x = \int_{\cdot}^{b} K(t,\cdot)\; x(t)\; dt \; .$$

Thus the normal equation $\mathfrak{B}^*\mathfrak{B}x = \mathfrak{B}^* y$ takes the form

$$\text{(3.11)} \qquad \int_s^b \int_a^t K(t, s)\, K(t, r)\, x(r)\; dr\; dt - \int_s^b K(t, s)\; y(t)\; dt = 0.$$

Interchanging the order of integration in the double integral in (3.11) leads to

$$\int_a^s \int_s^b K(t, s)\; K(t, r)\; x(r)\; dr\; dt + \int_s^b \int_r^b K(t, s)\; K(t, r)\; x(r)\; dt\; dr$$

$$- \int_s^b K(t, s)\; y(t)\; dt = 0 \; .$$

Define

$$N(s, t) = \int_t^b K(r, s)\; K(r, t)\; dr \; ,$$

and

$$z(s) = \int_s^b K(t, s)\; y(t)\; dt \; .$$ Then (3.11) takes

the form

$$\int_a^s N(s, t)\; x(t)\; dt \;\delta\; \int_s^b N(t, s)\; x(t)\; dt = z(s),$$

or

$$\text{(3.12)} \qquad \int_a^b \widetilde{N}(s, t)\; x(t)\; dt = z(s),$$

where

$$\widetilde{N}(s, t) = \begin{cases} N(s, t) & a \le t < s \\ N(t, s) & s \le t < b. \end{cases}$$

Thus the problem of finding the best approximate solution of the Volterra integral equation (3.5) is tantamount to finding the solution of minimal norm of the Fredholm integral equation (3.12).

Finally the problem of solving (3.6) in the least squares sense is equivalent to solving an equation of the form (3.9) with $\mathfrak{K}$ replaced by $\mathfrak{B}$, or

$$(3.13) \quad x(s) - \lambda\int_a^s K(s, t)\, x(t)\, dt - \lambda\int_s^b K(t, s)\, x(t)\, dt$$
$$+ \lambda^2\int_s^b\int_a^t K(t, s)\, K(t, r)\, x(r)\, dr\, dt = y(s) - \lambda\int_s^b K(t, s)\, y(t)\, dt.$$

This is equivalent to the Fredholm equation of the second kind

$$(3.14) \quad x(s) - \lambda\int_a^b \widetilde{k}(s, t)\, x(t)\, dt = \widetilde{g}(s)$$

where

$$\widetilde{k}(s, t) = \widetilde{K}(s, t) - \lambda\widetilde{N}(s, t),$$

$$\widetilde{K}(s, t) = \begin{cases} K(s, t) & a \le t < s \\ K(t, s) & s < t \le b \end{cases}$$

and $\widetilde{N}(s, t)$ as before.

As corollaries to Theorem 2.1, we obtain existence and uniqueness theorems for the best approximate solution of minimal norm of the integral equations (3.4) and (3.6) for any $y \in L_2[a, b]$, and of the integral equations (3.3) and (3.5) for any $y \in R(\mathfrak{K}) \oplus R(\mathfrak{K})^{\perp}$ and $y \in R(\mathfrak{B}) \oplus R(\mathfrak{B})^{\perp}$ respectively.

Theorem 3.1. Let $K(s, t) \in L_2\{[a, b] \times [a, b]\}$ and λ be a characteristic value of the Fredholm integral operator (3.1), i.e., for some $\varphi \neq 0$, $\lambda\int_a^b K(s, t)\, \varphi(t)dt = \varphi(s)$. Then the generalized inverse of $(I - \lambda\mathfrak{K})$ exists and is bounded. Furthermore,

(a) if for a given $y \in L_2[a, b]$, (3.4) is solvable, i.e., $y \in N(I - \bar{\lambda}\mathfrak{K}^*)^{\perp}$, then $x^* = (I - \lambda\mathfrak{K})^{\dagger}y$ is the unique solution with minimal L_2-norm;

(b) if for a given $y \in L_2[a, b]$, (3.4) does not have a solution, i.e., $y \notin N(I - \bar{\lambda}\mathfrak{K}^*)^{\perp}$, then $x^* = (I - \lambda\mathfrak{K})^{\dagger}y$ is the best approximate solution of (3.4) of minimal norm, i.e., it minimizes $\|(I - \lambda\mathfrak{K})x - y\|$ and has a smaller L_2-norm than any other u that minimizes $\|(I - \lambda\mathfrak{K})x - y\|$.

Equivalently, the normal operator equation (3.9) is always solvable and $x^* = (I - \lambda\mathfrak{K})^{\dagger}y$ is the unique solution of minimal norm.

Similar results hold for the Volterra equation of the second kind (3.6), and the corresponding normal equation (3.14).

Theorem 3.2. Let $K(s, t) \in L_2\{[a, b] \times [a, b]\}$.

(a) The generalized inverse of $\mathfrak{K}$ exists on the domain $D(\mathfrak{K}^{\dagger}) = R(\mathfrak{K}) + R(\mathfrak{K})^{\perp}$. $\mathfrak{K}^{\dagger}$ is unbounded unless $R(\mathfrak{K})$ is finite dimensional.

(b) If for a given $y \in L_2[a, b]$, (3.3) is solvable in $L_2[a, b]$, i.e., if $y \in N(\mathfrak{K}^*)^{\perp}$ and $\sum\mu_n \,|\langle y, \varphi_n\rangle|^2 < \infty$, where $\{\mu_n\}$ are the characteristic values and $\{\varphi_n\}$ are the orthonormal eigenvectors of the operator $\mathfrak{K}\mathfrak{K}^*$, then $\mathfrak{K}^{\dagger}y$ is the unique solution with minimal L_2-norm.

(c) If for a given element $y \in L_2[a, b]$, (3.3) does not have a solution,

then $x^* = \mathfrak{K}^\dagger y$ is the unique best approximate solution of (3.3) for each $y \in R(\mathfrak{K}) + R(\mathfrak{K})^\perp$. The best approximate solution does not depend continuously on y unless $R(\mathfrak{K})$ is finite dimensional.

Similar statements hold for the operator $\mathfrak{D}$ and (3.5).

As stated in Remark 2.3, the generalized inverse of a bounded linear operator T on a Banach space X to a Banach space Y, which can be defined relative to two projectors P and Q, does not possess the least squares property stated in Proposition 2.2. Thus Theorems 3.1 and 3.2 do not extend to integral equations on the Banach space $C[a, b]$ with the best approximate solution taken in the sense of the maximum norm! However, we can still study integral equations on $C[a, b]$ with the best approximate solution taken in the sense of the L_2-norm. The generalized inverse in the setting of $L_2[a, b]$ still gives a best approximate solution. Furthermore, the following theorem shows that for each $y \in C[a, b]$, $(I - \lambda\mathfrak{K})^\dagger y$ is a continuous function, i.e., whereas $(I - \lambda\mathfrak{K})^\dagger: L_2[a, b] \to L_2[a, b]$, the restriction of $(I - \lambda\mathfrak{K})^\dagger$ to $C[a, b]$ has its range in $C[a, b]$.

Theorem 3.3. $(I - \lambda\mathfrak{K})^\dagger y \in C[a, b]$ for any $y \in C[a, b]$ and $K(s, t) \in C\{[a, b] \times [a, b]\}$.

Proof. For $K(s, t) \in C\{[a, b] \times [a, b]\}$, consider $I - \lambda\mathfrak{K}$. Clearly $\mathfrak{K} : L_2[a, b] \to C[a, b]$ and $R(I - \lambda\mathfrak{K})$ is closed in $L_2[a, b]$. Hence

$$L_2[a, b] = R(I - \lambda\mathfrak{K}) \oplus \{R(I - \lambda\mathfrak{K})\}^\perp.$$

$$= R(I - \lambda\mathfrak{K}) \oplus N(I - \bar{\lambda}\mathfrak{K}^*).$$

The given continuous function y can be written as $y = y_1 + y_2$, where $y_1 = (I - \lambda\mathfrak{K})x$

for some $x \in L_2[a, b]$ and $(I - \overline{\lambda}\mathfrak{K}^*)y_2 = 0$, i.e., $y_2 = \overline{\lambda}\mathfrak{K}^* y_2$ which implies that y_2 is a continuous function. Thus $y_1 = y - y_2 = (I - \lambda\mathfrak{K})x$ is also continuous. Now $(I - \lambda\mathfrak{K})^{\dagger}(I - \lambda\mathfrak{K}) = P$, where P is the orthogonal projection on $N(I - \lambda\mathfrak{K})^{\perp}$, and

$$L_2[a, b] = N(I - \lambda\mathfrak{K}) \oplus N(I - \lambda\mathfrak{K})^{\perp}.$$

Thus we have $x = x_1 + x_2$, where $(I - \lambda\mathfrak{K})x_1 = 0$ and $x_2 \in N(I - \lambda\mathfrak{K})^{\perp}$. This shows that x_1 is also continuous and hence so is x_2. On the other hand $x_2 = (I - \lambda\mathfrak{K})^{\dagger}y$ since $(I - \lambda\mathfrak{K})x = y$ and

$$(I - \lambda\mathfrak{K})^{\dagger}(I - \lambda\mathfrak{K})\,x = Px = (I - \lambda\mathfrak{K})^{\dagger}y = x_2.$$

This completes the proof.

Remark. In view of Proposition 2.4d, the theory presented here also applies to integral equations with normal operators. For simplicity we limit our presentation to the integral operators described in Theorems 3.1 and 3.2. The modifications are simple in view of the excellent exposition on normal operators in the context of integral equations given in Zaanen [51].

4. A Generalization to Integral Equations of a Method of Cimmino and Related Aspects of the Successive Iterative Method for Best Approximate Solutions.

Cimmino [5] devised an iterative scheme for the solution of a linear system of equations $Ax = y$, where $A = (a_{ij})$ is a square matrix, which converges even if the system of equations is inconsistent, provided that the rank of the matrix A is greater than one. (See Problem 16, p. 119 in Householder [19]; Cimmino's method for matrices was examined in the setting of generalized inverses by Votruba [47].) In this section we generalize Cimmino's method to integral equations of the first kind and prove its convergence to the best approximate solution $\mathfrak{K}^{\dagger}y$, for any $y \in \mathcal{D}(\mathfrak{K}^{\dagger})$, provided $\mathcal{R}(\mathfrak{K})$ is of dimension greater than one. We also show that the generalization of Cimmino's method can be recast as a successive approximation method with a specifically chosen averaging parameter, and establish convergence theorems for best approximate solutions of integral equations of the first and second kinds.

To motivate the generalization and to place the method in proper perspective, we first discuss briefly Cimmino's method for matrices. Let $r_1, \ldots, r_n$ denote the rows of the matrix A; these rows determine n hyperplanes in R_n given by

$$H_i = \{x : \langle r_i, x\rangle = y_i\}, \; i = 1, \ldots, n.$$

Let $x^{(o)}$ be an initial approximation. We place a mass $m_j > 0$ at the reflection of the point $x^{(o)}$ with respect to the hyperplane H_j, $j = 1, \ldots, n$. For the next iteration we take the centroid of the system of n masses, and continue the iterations in this fashion using the same respective masses m_j. Algebraically this algorithm can be written in the form

$$x^{(k)} = \left(\sum_{j=1}^{n} m_j\right)^{-1} \sum_{i=1}^{n} m_i \left\{x^{(k-1)} + 2\,\frac{y_i - \langle r_i,\, x^{(k-1)}\rangle}{\|r_i\|^2}\, r_i\right\}$$

$$= x^{(k-1)} - \frac{2}{\mu} \sum_{i=1}^{n} m_i \left(\frac{\langle r_i,\, x^{(k-1)}\rangle - y_i}{\|r_i\|^2}\right) r_i \ ,$$

where $\mu = \sum_{k=1}^{n} m_k$. Setting $B = \left(\frac{\delta_{ij}}{\|r_i\|}\right)_{n\times n}$, $W = (\delta_{ij} m_i)_{n\times n}$, where δ_{ij} is the Kronecker delta, we have

$$x^{(k)} = x^{(k-1)} - \frac{2}{\mu} A^* W B [A\, x^{(k-1)} - y]$$

$$= [I - \frac{2}{\mu} A^* W B A]\, x^{(k-1)} + \frac{2}{\mu} A^* W B y.$$

Votruba [47] has shown that if rank $A > 1$ and $m_i = \|r_i\|^2$, $1 \le i \le n$, then the sequence $\{x^{(k)}\}$ converges to $(I - P)\, x^{(o)} + A^{\dagger} y$, where P is the orthogonal projection on $N(A)^{\perp}$.

One can extend Cimmino's method to Fredholm equations of the first kind

$$(4.1) \qquad \mathfrak{K}x = \int_a^b K(\cdot,\, t)\, x(t)\, dt = y \ , \qquad y \in L_2[a,\, b],$$

when $K(s,\, t) \in L_2\{[a,\, b] \times [a,\, b]\}$, by defining the family of hyperplanes

$$H_s = \{x \in L_2[a,\, b] : \int_a^b K(s,\, t)\, x(t)\, dt = y(s)\}$$

for almost every $s \in [a, b]$. Then the orthogonal projection of a function $x_0 \in L_2[a, b]$ into the hyperplane H_s is given by the function

$$(4.2) \qquad z_s = x_0 + \lambda(s)\, K(s, \cdot)\ ,$$

where

$$\lambda(s) = \frac{y(s) - \int_a^b K(s, r)\, x_0(r)\, dr}{\int_a^b |K(s, r)|^2\, dr}\ ,$$

and the reflection of x_0 through H_s is given by $x_0 + 2\lambda(s)\, K(s, \cdot)$.

We first note that $z_s \in H_s$. Indeed,

$$\int_a^b K(s, t)\, z_s(t)\, dt = \int_a^b K(s, t)\, x_0(t)\, dt + \lambda(s) \int_a^b K(s, t)\, K(s, t)\, dt$$

$$= \int_a^b K(s, t)\, x_0(t)\, dt + \frac{y(s) - \int_a^b K(s, r)\, x_0(r)\, dr}{\int_a^b |K(s, r)|^2 dr} \int_a^b |K(s, t)|^2 dt$$

$$= y(s).$$

Thus for almost all $s \in [a, b]$, $z_s \in H_s$. Now we show that $x_0 - z_s$ is orthogonal to $z_s - z$ for all $z \in H_s$

$$\langle x_0 - z_s,\, z_s - z\rangle = \langle -\lambda(s)\, K(s,\cdot),\, x_0 + \lambda(s)\, K(s,\cdot) - z\rangle$$

$$= -\lambda(s) \{\langle K(s, \cdot), x_0\rangle + \frac{\bar{y} - \int_a^b K(s, r)\, x_0(r)\, dr}{\int_a^b |K(s, r)|^2 dr} \int_a^b |K(s, r)|^2 dr$$

$$- \int_a^b K(s, r)\, \overline{z(r)}\, dr\} = 0 \text{ for } z \in H_s.$$

The next iterate x_1 in the Cimmino iteration would be the centroid of the family of points with the appropriate weight functions. That is, letting $m(s) = \int_a^b |K(s, t)|^2 \, dt$ be the mass density and $\beta = \int_a^b\int_a^b |K(s, t)|^2 \, dt \, ds$ be the total mass, we get

$$x_1(t) = \frac{1}{\beta} \int_a^b m(s) \, [x_0(t) + 2 \, \lambda(s) \, K(s, t)] \, ds$$

$$= x_0(t) - \frac{2}{\beta} \int_a^b K(s, t) \int_a^b K(s, r) \, x_0(r) \, dr \, ds$$

$$+ \frac{2}{\beta} \int_a^b K(s, t) \, y(s) \, ds,$$

that is,

$$x_1(t) = (I - \frac{2}{\beta} \mathfrak{K}^*\mathfrak{K}) \, x_0(t) + \frac{2}{\beta} (\mathfrak{K}^* y)(t) \, ,$$

and in general,

$$(4.3) \qquad x_{n+1}(t) = (I - \frac{2}{\beta} \mathfrak{K}^* \mathfrak{K}) \, x_n(t) + \frac{2}{\beta} (\mathfrak{K}^* y)(t) \, .$$

We note that (4.3) is a particular realization of the successive approximation scheme with an averaging parameter,

$$(4.4) \qquad x_{n+1} = x_n - \alpha \, \mathfrak{K}^*\mathfrak{K} x_n + \alpha \, \mathfrak{K}^* y$$

for a solution of the normal equation $\mathfrak{K}^*\mathfrak{K}x = \mathfrak{K}^*y$, or equivalently, a least squares solution of (4.1) if such a solution exists. The parameter α is a

prescribed number in this case. It should be pointed out, however, that one cannot apply the convergence theorems for iterative methods for singular linear operator equations developed by Keller [26], Petryshyn [35], and Votruba [47] since the range of $\mathcal{R}$ is not closed unless $\mathcal{R}$ is degenerate. For the same reason, (4.1) need not have a best approximate solution for each $y \in L_2[a, b]$. We shall show that for each $y \in \mathcal{D}(\mathcal{R}^\dagger)$, the sequence (4.3) converges to the best approximate solution of (4.1), provided $\dim R(\mathcal{R}) > 1$. The proof relies on a series representation for the generalized inverse of a bounded linear operator with arbitrary range and on the norm inequality given in Proposition 4.3. We now digress to discuss such representations and convergence of the successive approximation method to best approximate solutions of linear integral equations.

Let $T : X \to Y$ be a bounded linear operator and X, Y be two Hilbert spaces. Assume that $N(T) \neq \{0\}$ and let α be any positive real number. Then $(I - \alpha\, T^*T)^n$ converges to $I - P$, (P is the projection on $N(T)^\perp$) in the operator norm if and only if $R(T)$ is closed and $0 < \alpha < \frac{2}{\|T\|^2}$ (see Petryshyn [35]).

The optimal value of α is $\alpha_b = \frac{2}{\gamma^2+\|T\|^2}$, where

$$\gamma = \text{g.l.b}\ \left\{ \frac{\|Tx\|}{\|x\|} : x \in N(T)^\perp,\ x \neq 0 \right\} = \frac{1}{\|T^\dagger\|}$$

with the error estimate

$$\|(I - \alpha_b\, T^*T)^n - (I - P)\| \le \left\{ \frac{\|T\|^2-\gamma^2}{\|T\|^2+\gamma^2} \right\}^n = \left(\frac{c^2 - 1}{c^2 + 1}\right)^n$$

where $c = \|T\|\ \|T^\dagger\|$ is the pseudo-conditional number of T.

Furthermore, if T is closed, one easily obtains a Neumann-type series expansion for $T^\dagger$,

$$(4.5) \qquad T^\dagger = \sum_{n=0}^{\infty} (I - \alpha T^*T)^n \alpha T^*, \; 0 < \alpha < \frac{2}{\|T\|^2},$$

(see [34], [47], [2]).

We consider the successive approximations

$$(4.6) \qquad x_n = (I - \alpha T^*T) x_{n-1} + \alpha T^* y .$$

Using the relation $T^*TT^\dagger y = T^*Qy = T^*y$, it follows by recursion that

$$\begin{aligned} x_n - T^\dagger y &= (I - \alpha T^*T) x_{n-1} + \alpha T^*TT^\dagger y - T^\dagger y \\ &= (I - \alpha T^*T) (x_{n-1} - T^\dagger y) \\ &= (I - \alpha T^*T)^n (x_o - T^\dagger y). \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{n\to\infty} (x_n - T^\dagger y) &= \lim_{n\to\infty} (I - \alpha T^*T)^n (x_o - T^\dagger y) \\ &= (I - P) (x_o - T^\dagger y) = (I - P) x_o, \end{aligned}$$

since $T^\dagger y \in R(T^*)$. Thus we have

Proposition 4.1. For $0 < \alpha < \frac{2}{\|T\|^2}$, the sequence $\{x_n\}$ defined by (4.6) converges, for any initial approximation x_o, to $T^\dagger y + (I - P) x_o$, which is a best approximate solution of $Tx = y$. In particular the optimal choice $\alpha = \alpha_b$ yields

$$\|T^\dagger y + (I - P)\, x_o - x_n\| \le \left\{ \frac{\|T\|^2 - \gamma^2}{\|T\|^2 + \gamma^2} \right\}^n \|x_o - T^\dagger y\|.$$

Corollary 4.1. Let $\mathfrak{K}$ be a completely continuous operator on a Hilbert space H into H, and $\lambda \neq 0$ be a eigenvalue of $\mathfrak{K}$. Then the sequence (4.6) with $T = \mathfrak{K} - \lambda I$ and $0 < \alpha < \frac{2}{\|T\|^2}$ converges to a best approximate solution of the linear equation of the second kind (3.4).

Since convergence of $(I - \alpha T^*T)^n$ in the operator norm to $I - P$ is equivalent to the range of T being closed, and since pointwise convergence of $(I - \alpha T^*T)^n$ would be adequate to establish the convergence of $\{x_n\}$, it is natural to seek conditions under which $(I - \alpha T^*T)^n$ converges pointwise when $R(T)$ is not closed. For each $x \in X$, we have

$$\lim_{n\to\infty} (I - \alpha T^*T)^n x = (I - P)x, \text{ for } 0 < \alpha < \frac{2}{\|T\|^2},$$

where $R(T)$ is not necessarily closed. (See Showalter and Ben-Israel [41].) The series

$$\sum_{k=0}^{m} (I - \alpha T^*T)^k \alpha T^* y \qquad \text{for} \qquad 0 < \alpha < \frac{2}{\|T\|^2}$$

converges in norm monotonically to $T^\dagger y$ for any $y \in D(T^\dagger) = R(T) + R(T^\dagger)$. (Compare with (4.5) for the case of a closed range.) Moreover, if $Qy \in R(TT^*)$, then

$$\|T^\dagger y - \sum_{k=0}^{n} (I - \alpha T^*T)^k \alpha T^* y\|^2 \le \frac{\|T^\dagger y\|^2 \, \|(TT^*)^\dagger y\|^2}{\|(TT^*)^\dagger y\|^2 + n\alpha(2 - \alpha\|T\|^2)\, \|T^\dagger y\|^2} .$$

Rephrased in the setting of the iterative process (4.6), the above expansions yield easily the following proposition.

Proposition 4.2. Let X and Y be two Hilbert spaces over the same field and T be a bounded linear operator on X into Y, with the range of T not necessarily closed. The sequence (4.6) starting with $x_o = 0$, converges in norm monotonically to $T^{\dagger}y$ whenever $y \in \mathcal{D}(T^{\dagger}) = \mathcal{R}(T) + \mathcal{R}(T)^{\perp}$ and α is any fixed number in the range $0 < \alpha < \frac{2}{\|T\|^2}$. Moreover, if $Qy \in \mathcal{R}(TT^*)$, then

$$\|x_n - T^{\dagger}y\|^2 \leq \frac{\|T^{\dagger}y\|^2 \; \|(TT^*)^{\dagger}y\|^2}{\|(TT^*)^{\dagger}y\|^2 + n\,\alpha\,(2 - \alpha\|T\|^2)\|\,T^{\dagger}y\|^2} \; .$$

We now return to the consideration of convergence of the generalization of Cimmino's method for integral equations of the first kind.

Proposition 4.3. Let $\mathfrak{K}$ be the integral operator defined by (4.1), where K(s, t) $\in L_2\{[a, b] \times [a, b]\}$. If the dimension of the range of $\mathfrak{K}$ is greater than one, then

$$\|\mathfrak{K}\|^2 < \int_a^b \int_a^b |K(s, t)|^2 \, ds \, dt.$$

Proof. We will first show that dim $\mathcal{R}(\mathfrak{K}) > 1$ implies dim $\mathcal{R}(\mathfrak{K}^*\mathfrak{K}) > 1$. For if dim $\mathcal{R}(\mathfrak{K}^*\mathfrak{K}) \leq 1$, then

$$\mathcal{R}(\mathfrak{K}^*\mathfrak{K})^{\perp} = \mathcal{N}(\mathfrak{K}^*\mathfrak{K}) = \mathcal{N}(\mathfrak{K}).$$

Thus the deficiency of $N(\mathfrak{K})$ is not greater than one and dim $R(\mathfrak{K}) \leq 1$.

It is well known that

$$\rho(\mathfrak{K}^*\mathfrak{K}) = \|\mathfrak{K}^*\mathfrak{K}\| = \|\mathfrak{K}\|^2 \leq \mu \tag{4.7}$$

where $\rho(\mathfrak{K}^*\mathfrak{K})$ denotes the spectral radius of $\mathfrak{K}^*\mathfrak{K}$. (See, for instance, [43].) Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ be the eigenvalues of $\mathfrak{K}^*\mathfrak{K}$ with a corresponding orthonormal set of eigenfunctions $\phi_1, \phi_2, \ldots,$ and let $M(s, t) = \int_a^b K(u, s)\, K(u, t)\, du$. Then by Bessel's inequality,

$$\lambda_1^2\, \phi_1^2(s) = \int_a^b M(s, t)\, \phi_1(t)\, dt < \int_a^b [M(s, t)]^2 dt.$$

Note that the strict inequality in Bessel's inequality holds here since $R(\mathfrak{K}^*\mathfrak{K}) > 1$. Integrating both sides of the preceding inequality, we get

$$\lambda_1^2 < \int_a^b \int_a^b [M(s, t)]^2 dt\, ds = \int_a^b \int_a^b \left[\int_a^b K(u, s)\, K(u, t)\, du\right]^2 dt\, ds$$

$$\leq \int_a^b \int_a^b \left\{\int_a^b [K(u, s)]^2\, du \int_a^b [K(u, t)]^2\, du\right\} dt\, ds$$

$$= \left\{\int_a^b \int_a^b [K(u, s)]^2\, du\, ds\right\}^2 = \beta^2 .$$

Thus $\lambda_1 < \beta$. Thus from (4.7) we have $\|\mathfrak{K}\|^2 < \beta$ since $\lambda_1 = \rho(\mathfrak{K}^*\mathfrak{K})$.

Theorem 4.1. If dim $R(\mathfrak{K}) > 1$, then the generalized method of Cimmino converges monotonically to a best approximate solution of minimal norm of the integral equation of the first kind, starting from the initial approximation $x_0 = 0$, for

any $y \in \mathcal{D}(\mathfrak{K}^\dagger) = \mathcal{R}(\mathfrak{K}) + \mathcal{R}(\mathfrak{K})^\perp$ and

$$\|x_n - \mathfrak{K}^\dagger y\|^2 \le \frac{\|\mathfrak{K}^\dagger y\|^2 \ \|(\mathfrak{K}\mathfrak{K}^*)^\dagger y\|^2}{\|(\mathfrak{K}\mathfrak{K}^*)^\dagger y\|^2 + \frac{4n}{\beta^2}(\beta - \|\mathfrak{K}\|^2)\ \|\mathfrak{K}^\dagger y\|^2},$$

where $\beta = \int_a^b \int_a^b [K(u, s)]^2 \, du \, ds$.

Proof. The theorem is a consequence of Propositions 4.2 and 4.3.

Let $K(s, t)$ be a symmetric positive definite kernel and assume that the integral equation (3.1) is solvable. Fridman [11] has shown that for any α in the interval $0 < \alpha < 2\alpha_1$, where α_1 is the smallest characteristic value of the kernel $K(s, t)$, then the sequence

$$x_{n+1}(s) = x_n(s) + \alpha[y(s) - \mathfrak{K}x_n(t)]$$

converges in the norm of $L_2[a, b]$ to the solution starting from any initial approximation $x_o \in L_2[a, b]$. (See also Mikhlin and Smolitskiy [31].) Bialy [4] generalized Fridman's result and proved the following theorem. Let A be a bounded linear operator on a Hilbert space H into H, and suppose also that A is nonnegative: $\langle Ax, x\rangle \ge 0$ for all $x \in H$. For $y \in H$, $x_o \in H$, consider the iterative process

$$x_{n+1} = x_n + \alpha(y - Ax_{n-1})$$

where $0 < \alpha < 2\ \|A\|^{-1}$. Then $Ax_n \to Qy$, where Q is the orthogonal projection on

$R(A)$. $\{x_n\}$ converges if and only if the equation $Ax = y$ has a solution, in which case $x_n \to (I - P)x_o + \hat{x}$, where $\hat{x}$ is the solution of minimal norm. Related results on iterative methods for the solutions of nonnegative operators were obtained by Keller [26]. Proposition 4.2 generalizes the results of Fridman and Bialy to the setting of best approximate solutions and expresses the results and error bounds in terms of generalized inverses.

5. Steepest Descent and Conjugate Gradient Methods for Best Approximate Solutions of Linear Integral Equations.

We consider first the integral equation of the second kind (3.4). Kantorovich [24] has shown that if K(s, t) is symmetric and $\lambda < \lambda_k$, $k = 1, 2, \ldots$, where λ_k are the characteristic values of the kernel K(s, t), then the method of steepest descent for the solution of (3.4), i.e., the sequence

$$x_{n+1}(s) = x_n(s) + \alpha_n z_n(s)$$

where

$$z_n(s) = Lx_n(s) \equiv x_n(s) - \lambda\int_a^b K(s, t)\, x_n(t)\, dt - y(s),$$

$$\alpha_n = -\frac{\int_a^b [Lx_n(s)]^2 ds}{Q(L(x_n))}$$

and

$$Q(u) = \int_a^b u^2(s)\, ds - \lambda\int_a^b \int_a^b K(s, t)\, u(s)\, u(t)\, ds\, dt$$

converges to the unique solution x^* of (3.4). The speed of convergence is determined by

$$\|x_n - x^*\| \le \sqrt{\frac{c}{1-m_1}} \left(\frac{m_2-m_1}{2-m_1-m_2}\right)^n$$

where $m_1 = \max_k \frac{\lambda}{\lambda_k}$, $m_2 = \min_k \frac{\lambda}{\lambda_k}$ and c is a constant. This result follows from a direct application of the well known theorem of Kantorovich [24, 25] on the

convergence of the method of steepest descent for positive definite bounded linear operators on a Hilbert space. See also Hayes [52] for related results.

When $m_1 = 1$, (3.4) of course may not have a solution. However, if a solution exists, then the sequence of steepest descent converges to it. Kantorovich's theorem does not apply to integral equations of the first kind with nondegenerate kernel.

In this section we extend the applicability of the method of steepest descent and the conjugate gradient methods to integral equations with nonunique solutions and to integral equations of the first kind. Convergence of these methods in the mentioned settings will be asserted using recent results of the authors [22], [23], [32] on singular linear operator equations.

Let T be a bounded linear operator on a Hilbert space X into a Hilbert space Y. The method of steepest descent for minimizing the functional $J(x) = \|Tx - y\|^2$ for $y \in Y$ is defined by the following sequence starting with an initial approximation x_o

$$x_{n+1} = x_n - \alpha_n r_n \qquad n = 0, 1, \ldots \tag{5.1}$$

where

$$r_n = T^*(Tx_n - y) \tag{5.2}$$

and

$$\alpha_n = \frac{\|r_n\|^2}{\|Tr_n\|^2} . \tag{5.3}$$

The following theorem is an immediate consequence of [32] using (3.9) - (3.10) and Theorem 3.1.

Theorem 5.1. Let $T = I - \lambda\mathfrak{K}$, where $\mathfrak{K}$ is as in Section 3. Then the sequence $\{x_n\}$ of steepest descent defined by (5.1) - (5.3) with any initial approximation $x_o \in L_2[a, b]$ converges in $L_2[a, b]$ to the best approximate solution $(I - \lambda\mathfrak{K})^{\dagger}y + (I - P)x_o$ of the integral equation of the second kind (3.4) for each $y \in L_2[a, b]$, and

$$\|(I - \lambda\mathfrak{K})^{\dagger}y + (I - P)x_o - x_n\| \le C\left[\frac{M-m}{M+m}\right]^n,$$

where C is a constant and

$$m\|x\|^2 \le \langle(I - \lambda\mathfrak{K})(I - \bar{\lambda}\mathfrak{K}^*)x,x\rangle \le M\|x\|^2,\ x \in R(I - \bar{\lambda}\mathfrak{K}^*).$$

The sequence of steepest descent can be written in this case in the form

$$x_{n+1}(s) = (1 - \alpha_n)\, x_n(s) - \alpha_n\, [\lambda\int_a^b k(s, t)\, x(t)\, dt - g(s)],$$

where k(s, t) and g(s) are defined in (3.10).

The sequence $\{x_n\}$ in the above theorem converges in the mean. However, a sequence may be constructed using $\{x_n\}$ that converges uniformly to $(I - \lambda\mathfrak{K})^{\dagger}y + (I - P)\, x_o$. Indeed, if we define

$$z_n(s) = g(s) + \lambda\int_a^b k(s, t)\, x_n(t)\, dt,$$

then $\{z_n(s)\}$ converges uniformly to a best approximate solution of (3.4).

For integral equations of the first kind (3.3) we have the following theorem as a consequence of Theorem 3.2 and Theorem 3.2 in [22].

Theorem 5.2. Let $T = \mathfrak{K}$ as defined in Section 3. If $Qy \in R(\mathfrak{K}\mathfrak{K}^*)$, then the sequence $\{x_n\}$ of steepest descent defined by (5.1) - (5.3) with initial approximation $x_o = 0$ converges monotonically to $\mathfrak{K}^\dagger y$, the best approximate solution of minimal norm of the integral equation of the first kind (3.3), and

$$\|x_n - \mathfrak{K}^\dagger y\|^2 \le \frac{\|\mathfrak{K}\|^2 \ \|\mathfrak{K}^\dagger y\|^2 \ \|(\mathfrak{K}\mathfrak{K}^*)^\dagger y\|^2}{\|\mathfrak{K}\|^2 \ \|(\mathfrak{K}\mathfrak{K}^*)^\dagger y\|^2 + n \ \|\mathfrak{K}^\dagger y\|^2}$$

for $n = 1, 2, \cdots$.

We now consider the conjugate gradient method of Hestenes and Stiefel [17], [52] for minimizing the functional $J(x) = \|Tx - y\|^2$. We let $r_o = p_o = T^* (Tx_o - y)$ and if $p_o \neq 0$, then compute $x_1 = x_o - \alpha_o p_o$, where $\alpha_o = \dfrac{\|r_o\|^2}{\|Tp_o\|^2}$. For $n = 1, 2, \ldots$ compute

(5.4) $$r_n = T^*(Tx_n - y) = r_{n-1} - \alpha_{n-1} T^*Tp_{n-1}$$

(5.5) $$\alpha_{n-1} = \frac{\langle r_{n-1}, p_{n-1}\rangle}{\|Tp_{n-1}\|^2}$$

(5.6) $$p_n = r_n + \beta_{n-1} p_{n-1}, \quad \beta_{n-1} = - \frac{\langle r_n, T^*Tp_{n-1}\rangle}{\|Tp_{n-1}\|^2}$$

$$x_{n+1} = x_n - \alpha_n p_n. \tag{5.7}$$

The following theorem is a consequence of Theorem 3.1, and Theorem 4.1 in [23].

Theorem 5.3. Let $T = I - \lambda \mathfrak{K}$ where $\mathfrak{K}$ is as in Section 3. Then the sequence $\{x_n\}$ generated by the conjugate gradient method (5.4) - (5.7) converges monotonically starting from any initial approximation $x_o \in L_2[a, b]$ to the best approximate solution $u = (I - \lambda\mathfrak{K})^{\dagger} y + (I - P)x_o$ of the integral equation of the second kind (3.4) for each $y \in L_2[a, b]$, and

$$\|x_n - u\|^2 \le \frac{\langle r_o , e_o \rangle}{m} \left[\frac{M - m}{M + m}\right]^{2n}, \; n = 1, 2, \ldots$$

where m and M are the spectral bounds of the restriction of the operator $(I - \bar{\lambda}\mathfrak{K}^*)(I - \lambda\mathfrak{K})$ to $R(I - \bar{\lambda}\mathfrak{K}^*)$.

As a consequence of Theorem 3.2, and Theorem 5.1 in [23] we have the following theorem for integral equations of the first kind.

Theorem 5.4. Let $T = \mathfrak{K}$ as defined in Section 3. If $Qy \in R(\mathfrak{K}\mathfrak{K}^*\mathfrak{K})$, then the conjugate gradient method (5.4) - (5.7) with initial approximation $x_o \in R(\mathfrak{K}^*\mathfrak{K})$, converges monotonically to the best approximate solution of minimal norm of the integral equation of the first kind (3.3) and

$$\|x_n - \mathfrak{K}^{\dagger} y\|^2 \le \frac{\|\mathfrak{K}\|^2 \; \|x_o - \mathfrak{K}^{\dagger} y\|^2 \; \|(\mathfrak{K}^*)^{\dagger} x_o - (\mathfrak{K}\mathfrak{K}^*)^{\dagger} y\|^2}{\|\mathfrak{K}\|^2 \; \|(\mathfrak{K}^*)^{\dagger} x_o - (\mathfrak{K}\mathfrak{K}^*)^{\dagger} y\|^2 + n\|x_o - \mathfrak{K}^{\dagger} y\|^2} \; .$$

The theorems on the convergence of the successive approximation method, the steepest descent and the conjugate gradient methods also apply to best approximate solutions of Volterra linear integral equations of the first and second kinds.

6. Bibliographical Comments.

There is an extensive literature on the steepest descent and the conjugate gradient methods for linear operator equations, going back to the papers of L. V. Kantorovich [24] and R. M. Hayes [52], respectively, and to the recent work of the authors [32], [22], [23] on singular operator equations. For various contributions to these methods for linear operator equations, we refer the reader to the bibliographical comments made in [22], [23]; the latter reference also contains an extensive bibliography on the conjugate gradient method and related variants for linear and nonlinear operator equations and minimization problems in various settings. In the present paper we have confined our bibliography to relevant references dealing with integral equations and related aspects of generalized inverses and iterative methods. For other contributions to generalized inverses of linear operators and related topics not considered here, see Nashed [53].

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AR 70-31 Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Mathematics Research Center, University of Wisconsin, Madison, Wis. 53706	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP None

3. REPORT TITLE

ITERATIVE METHODS FOR BEST APPROXIMATE SOLUTIONS OF LINEAR INTEGRAL EQUATIONS OF THE FIRST AND SECOND KINDS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Summary Report: no specific reporting period.

5. AUTHOR(S) (First name, middle initial, last name)

W. J. Kammerer and M. Z. Nashed

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1971	41	53

8a. CONTRACT OR GRANT NO. Contract No. DA-31-124-ARO-D-462	9a. ORIGINATOR'S REPORT NUMBER(S) #1117
b. PROJECT NO. None	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) None
d.	

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Army Research Office-Durham, N.C.

13. ABSTRACT

Least squares solutions of Fredholm and Volterra equations of the first and second kinds are studied using generalized inverses. The method of successive approximations, the steepest descent and the conjugate gradient methods are shown to converge to a least squares solution or to a least squares solution of minimal norm, both for integral equations of the first and second kinds.

An iterative method for matrices due to Cimmino is generalized to integral equations of the first kind and its convergence to the least squares solution of minimal norm is established.

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

AGO 5696A